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KAC-MOODY GROUPS AND THEIR REPRESENTATIONS

DMITRIY RUMYNIN

To Leonid Arkadievich Bokut with admiration

ABSTRACT. In this expository paper we review some recent results about representations of Kac-Moody groups. We sketch the construction of these groups. If practical, we present the ideas behind the proofs of theorems. At the end we pose open questions.

Kac-Moody Lie algebras are well-known generalisations of simple finite-dimensional Lie algebras, subject of 1533 research papers on MathSciNet and at least 3 beautiful monographs [Ka-85, Wa-91, Ct-05] Kac-Moody groups are less well-known cousins, subject of only 214 research papers on MathSciNet. One issue with them is that there are several different notions of a Kac-Moody group:

- a group valued functor on commutative rings defined by Tits, a generalisation of $R \mapsto \mathrm{SL}_n(R[z, z^{-1}])$,
- a locally compact totally disconnected group, a generalisation of $\mathrm{SL}_n(\mathbb{F}_q((z)))$,
- an ind-algebraic group, a generalisation of $\mathrm{SL}_n(\mathbb{C}((z)))$,
- a more complicated topological group, e.g., $\mathrm{SL}_n(\mathbb{Q}_p((z)))$.

In this survey, we review some new results about the first two types of Kac-Moody groups and their representations. We give examples and sketch proofs whenever it is practical. The only completely new results are in Section 1.4 where full proofs are given.

There are instructional sources about their Group Theory and Geometry [CaRé-09, Ma-13, Ré-02] but not about their Representation Theory. A reader interested in ind-algebraic Kac-Moody groups can consult a monograph [Ku-02] but someone who wants to learn about more complicated Kac-Moody groups will need to look at scholarly sources [Ro-06, Ro-16]. We start without further ado.

1. REPRESENTATIONS OF UNCOMPLETED GROUP

1.1. Kac-Moody Lie algebra. Let $\mathcal{A} = (A_{i,j})_{n \times n}$ be a square matrix with coefficients in a commutative ring \mathbb{K} . A *realisation* of \mathcal{A} is a collection $\mathcal{R} = (\mathfrak{h}, h_1, \dots, h_n, \alpha_1, \dots, \alpha_n)$ where \mathfrak{h} is a finitely generated free \mathbb{K} -module, h_i are \mathbb{K} -linearly independent elements of \mathfrak{h} , α_j are \mathbb{K} -linearly independent elements of \mathfrak{h}^* , and $\alpha_j(h_i) = A_{ij}$ for all i and j .

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A realisation gives several interesting Lie \mathbb{A} -algebras for any commutative \mathbb{K} -algebra \mathbb{A} . The first Lie \mathbb{A} -algebra is $\tilde{L}_{\mathcal{R}}(\mathbb{A})$: it is generated by $\mathfrak{h}_{\mathbb{A}} = \mathfrak{h} \otimes_{\mathbb{K}} \mathbb{A}$ and elements $e_1, \dots, e_n, f_1, \dots, f_n$ subject to the relations

$$[h, e_i] = \alpha_i(h)e_i, [h, f_j] = -\alpha_j(h)f_j, [h', h] = 0, [e_i, f_i] = h_i, [e_i, f_j] = 0 \text{ if } i \neq j$$

for all $h', h \in \mathfrak{h}_{\mathbb{A}}$. The Lie algebra $\tilde{L}_{\mathcal{R}}(\mathbb{A})$ is graded by the root group $X(\mathcal{R})$, the free abelian group generated by elements α_i . The grading is given by

$$\deg(h) = 0, \deg(e_i) = \alpha_i, \deg(f_i) = -\alpha_i.$$

Let $I_{\mathbb{A}}$ be the sum of all ideals of $\tilde{L}_{\mathcal{R}}(\mathbb{A})$, contained in the non-zero graded part $\bigoplus_{\gamma \neq 0} \tilde{L}_{\mathcal{R}}(\mathbb{A})_{\gamma}$. The second Lie algebra is $\hat{L}_{\mathcal{R}}(\mathbb{A}) := \tilde{L}_{\mathcal{R}}(\mathbb{A})/I_{\mathbb{A}}$ and the third Lie algebra is $L_{\mathcal{R}}(\mathbb{A}) := \hat{L}_{\mathcal{R}}(\mathbb{K}) \otimes_{\mathbb{K}} \mathbb{A}$. Although there is some literature on $L_{\mathcal{R}}(\mathbb{A})$ for a general \mathcal{A} [VKa-71], these algebras merit further investigation (cf. 3.1 and 3.2).

If \mathcal{A} is a generalised Cartan matrix, we set $\mathbb{K} = \mathbb{Z}$, call a realisation (over \mathbb{Z}) a *root datum* and denote it \mathcal{D} . While both $\hat{L}_{\mathcal{D}}(\mathbb{A})$ and $L_{\mathcal{D}}(\mathbb{A})$ deserve to be called *Kac-Moody algebras*, the actual definition of a Kac-Moody algebra is different. Let $\mathcal{U}_{\mathbb{Z}}$ be the divided powers integral form of the universal enveloping algebra $U(L_{\mathcal{D}}(\mathbb{C}))$. Then a *Kac-Moody algebra* is defined as

$$\mathfrak{g}_{\mathbb{Z}} := L_{\mathcal{D}}(\mathbb{C}) \cap \mathcal{U}_{\mathbb{Z}}, \quad \mathfrak{g}_{\mathbb{A}} := \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{A}.$$

It inherits a triangular decomposition $\mathfrak{g}_{\mathbb{A}} = (\mathfrak{n}_{-} \otimes \mathbb{A}) \oplus (\mathfrak{h} \otimes \mathbb{A}) \oplus (\mathfrak{n}_{+} \otimes \mathbb{A})$ from $\mathfrak{g}_{\mathbb{Z}} = \mathfrak{n}_{-} \oplus \mathfrak{h} \oplus \mathfrak{n}_{+}$ where $\mathfrak{n}_{- \mathbb{C}}$ is the Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$ generated by all f_i , \mathcal{U}_{-} is the divided powers \mathbb{Z} -form of $U(\mathfrak{n}_{- \mathbb{C}})$ and $\mathfrak{n}_{-} := \mathfrak{n}_{- \mathbb{C}} \cap \mathcal{U}_{-}$ (ditto for \mathfrak{n}_{+} using e_i -s and \mathcal{U}_{+}). If \mathbb{F} is a field of characteristic p , the Lie algebra $\mathfrak{g}_{\mathbb{F}}$ is restricted with the p -operation

$$(h \otimes 1)^{[p]} = h \otimes 1, \quad (x \otimes 1)^{[p]} = x^p \otimes 1 \quad \text{where } h \in \mathfrak{h}, x \in \mathfrak{n}_{\pm}$$

where x^p is calculated inside the associative \mathbb{Z} -algebra $\mathcal{U}_{\pm} \leq \mathcal{U}$ [Ma-13, Th. 4.39].

If $p > \max_{i \neq j} (-A_{i,j})$, then all the three Kac-Moody coincide: $\hat{L}_{\mathcal{D}}(\mathbb{F}) = L_{\mathcal{D}}(\mathbb{F}) = \mathfrak{g}_{\mathbb{F}}$ [Ro-16, Ma-13] but it is probably no longer true for small primes.

1.2. Kac-Moody group. The algebras $\mathcal{U}_{\mathbb{Z}}$ and $\mathfrak{g}_{\mathbb{Z}}$ inherit the grading by $X(\mathbb{R})$. It is also known as the root decomposition

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{\alpha \in \Phi \subseteq X(\mathcal{D})} \mathfrak{g}_{\mathbb{C} \alpha}.$$

The set of roots splits into two disjoint parts: real roots $\Phi^{re} := W\{\alpha_1, \dots, \alpha_n\}$ (where W is the Weyl group) and imaginary roots $\Phi^{im} := \Phi \setminus \Phi^{re}$.

The *Kac-Moody group* is a functor $G_{\mathcal{D}}$ from commutative rings to groups. Its value on a field \mathbb{F} can be described as

$$G_{\mathcal{D}}(\mathbb{F}) = T * \ast_{\alpha \in \Phi^{re}} U_{\alpha} / \langle \text{Tits' relations} \rangle, \quad T = \mathfrak{h} \otimes_{\mathbb{Z}} \mathbb{F}^{\times}, \quad U_{\alpha} \cong \mathbb{F}^{+},$$

where T is a torus and $U_{\alpha} = \{X_{\alpha}(\mathbf{t})\}$, $X_{\alpha}(\mathbf{t})X_{\alpha}(\mathbf{s}) = X_{\alpha}(\mathbf{t} + \mathbf{s})$ is a root subgroup. There are different ways to write Tits' relations: the reader should consult classical papers [CtCh-93, T-87] for succinct presentations. Note that Tits' Relations have infinitely many generators and relations unless \mathcal{A} is of finite type.

However, if the field $\mathbb{F} = \mathbb{F}_q$, $q = p^m$ is finite and under mild assumptions on \mathcal{A} , the groups $G_{\mathcal{D}}(\mathbb{F}_q)$ are finitely presented [AM-97] (cf. [CKR-16] for concrete finite presentations of affine groups) and simple [CaRé-09a]. Thus, the groups $G_{\mathcal{D}}(\mathbb{F}_q)$ form a good source of finitely-presented simple (non-linear) groups.

It is important for us that they have a BN-pair with $B = T \ltimes U_+$ where U_+ is the subgroup generated by all U_α for positive real roots α .

1.3. Adjoint representation. The group $G_{\mathcal{D}}(\mathbb{F})$ acts the Lie algebra $\mathfrak{g}_{\mathbb{F}}$ via adjoint action [Ma-13, Ré-02]. The torus action comes from the $X(\mathcal{D})$ -grading

$$\mathrm{Ad}(h \otimes \mathbf{t})(a) = \mathbf{t}^{\alpha(h)} a \quad \text{where } a \in (\mathfrak{g}_{\mathbb{F}})_{\alpha}$$

and the action of U_{α} is exponential:

$$\mathrm{Ad}(X_{\alpha}(\mathbf{t}))(a) = \mathbf{e}^{\mathrm{ad}(\mathbf{t}e_{\alpha})}(a) = \sum_{n=0}^{\infty} \mathbf{t}^n \mathrm{ad}(e_{\alpha}^{(n)})(a)$$

where e_{α} (rather than $e_{\alpha} \otimes 1$) is a non-zero element of $\mathfrak{g}_{\mathbb{F}\alpha}$ and $e_{\alpha}^{(n)} \in \mathcal{U} \otimes \mathbb{F}$ is its divided power. Notice that $\mathfrak{g}_{\mathbb{F}\alpha}$ is one-dimensional for a real root α .

We denote the image of Ad by $G_{\mathcal{D}}^{\mathrm{ad}}(\mathbb{F})$.

1.4. Over-restricted representations. Let \mathbb{F} be a field of positive characteristic p in this section. A representation (V, ρ) of the Lie algebra $\mathfrak{g}_{\mathbb{F}}$ is called *restricted* if $\rho(x)^p = \rho(x^{[p]})$ for all $x \in \mathfrak{g}_{\mathbb{F}}$. Each real root α yields an additive family of linear operators on a restricted representation

$$Y_{\alpha}(\mathbf{t}) := \mathbf{e}^{\rho(e_{\alpha})} = \sum_{k=0}^{p-1} \frac{1}{k!} \rho(e_{\alpha})^k.$$

These operators do not define an action of $G_{\mathcal{D}}(\mathbb{F})$ in general. The concept of an *over-restricted* representation, proposed recently to integrate representations from Lie algebras to algebraic groups [RWs-18], proves beneficial here as well. We say that a restricted representation (V, ρ) of $\mathfrak{g}_{\mathbb{F}}$ is *over-restricted* if $\rho(e_{\alpha})^{[(p+1)/2]} = 0$ for any real root α .

Proposition 1.1. (cf. [RWs-18]) *Suppose that (V, ρ) is an over-restricted representation $\mathfrak{g}_{\mathbb{F}}$. If $\mathrm{ad}(e_{\alpha}^{(p)})(x) = 0$ for some $x \in \mathfrak{g}_{\mathbb{F}}$, then*

$$(2) \quad \rho(\mathrm{Ad}(X_{\alpha}(\mathbf{t}))(x)) = Y_{\alpha}(\mathbf{t})\rho(x)Y_{\alpha}(-\mathbf{t}).$$

Proof. Observe by induction that for each $k = 1, 2, \dots, p-1$

$$(2) \quad \rho\left(\frac{1}{k!} \mathrm{ad}(e_{\alpha})^k(x)\right) = \sum_{j=0}^k \frac{(-1)^j}{(k-j)!j!} \rho(e_{\alpha})^{k-j} \rho(x) \rho(e_{\alpha})^j.$$

The condition $\mathrm{ad}(e_{\alpha}^{(p)})(x) = 0$ implies that $\mathrm{ad}(e_{\alpha}^{(n)})(x) = 0$ for all $n \geq p$ and $\rho(\mathrm{Ad}(X_{\alpha}(\mathbf{t}))(x)) = \sum_{k=0}^{p-1} \rho\left(\frac{1}{k!} \mathrm{ad}(\mathbf{t}e_{\alpha})^k(x)\right)$ stops at degree $p-1$. Using Formula (2), this is equal to

$$\sum_{i+j=0}^{p-1} \frac{(-1)^j}{i!j!} \rho(\mathbf{t}e_{\alpha})^i \rho(x) \rho(\mathbf{t}e_{\alpha})^j \spadesuit \sum_{i,j=0}^{p-1} \frac{(-1)^j}{i!j!} \rho(\mathbf{t}e_{\alpha})^i \rho(x) \rho(\mathbf{t}e_{\alpha})^j = \mathbf{e}^{\rho(\mathbf{t}e_{\alpha})} \rho(x) \mathbf{e}^{-\rho(\mathbf{t}e_{\alpha})},$$

exactly the right hand side. Notice that Equality \spadesuit holds because (V, ρ) is over-restricted: terms on the right, missing from the left, are all zero. \square

Consider an $X(\mathcal{D})$ -graded restricted representation (V, ρ) of \mathfrak{g} . Grading gives an action of T on V by $\hat{\rho}(h \otimes \mathbf{t})(v_{\alpha}) = \mathbf{t}^{\alpha(h)} v_{\alpha}$. An analogue of Proposition 1.1 holds for T :

$$(3) \quad \rho(\mathrm{Ad}(h \otimes \mathbf{t})(x)) = \hat{\rho}(h \otimes \mathbf{t})\rho(x)\hat{\rho}(h \otimes \mathbf{t}^{-1}).$$

Let G_V be the subgroup of $\mathrm{GL}(V)$ generated by $\widehat{\rho}(T)$ and all $Y_\alpha(\mathbf{t})$.

Theorem 1.2. *Suppose that $p > \max_{i \neq j}(-A_{i,j})$. If (V, ρ) is an $X(\mathcal{D})$ -graded over-restricted representations of $\mathfrak{g}_{\mathbb{F}}$, faithful on both T and $\mathfrak{g}_{\mathbb{F}}$, then*

$$\phi : G_V \rightarrow G_{\mathcal{D}}^{ad}(\mathbb{F}), \quad \phi(Y_\alpha(\mathbf{s})) = X_\alpha(\mathbf{s}), \quad \phi(\widehat{\rho}(t)) = t \quad \text{for } t \in T$$

is a surjective homomorphism of groups whose kernel is central and consists of $\mathfrak{g}_{\mathbb{F}}$ -automorphisms of V .

Proof. Let H be the free product of T and all additive groups U_α , $\alpha \in \Phi^{re}$. Both G_V and $G_{\mathcal{D}}^{ad}(\mathbb{F})$ are naturally quotients of H . If $x_1 * \dots * x_n \in \ker(H \rightarrow G_V)$ where all x_i are from the constituent groups then

$$\phi(x_1)\phi(x_2)\dots\phi(x_n) = I_V.$$

Formulas (1) and (3) imply that

$$\rho([\mathrm{Ad}(x_1)\mathrm{Ad}(x_2)\dots\mathrm{Ad}(x_n)](e_\beta)) = \rho(e_\beta), \quad \rho([\mathrm{Ad}(x_1)\mathrm{Ad}(x_2)\dots\mathrm{Ad}(x_n)](h)) = \rho(h)$$

for all $h \in \mathfrak{h}_{\mathbb{F}}$ and real roots β . Our restriction on p imply that $\mathfrak{g}_{\mathbb{F}}$ is generated by $\mathfrak{h}_{\mathbb{F}}$ and all e_β [Ro-16]. Consequently,

$$\rho([\mathrm{Ad}(x_1)\mathrm{Ad}(x_2)\dots\mathrm{Ad}(x_n)](y)) = \rho(y)$$

for all $y \in \mathfrak{g}_{\mathbb{F}}$. Since ρ is injective it follows that $[\mathrm{Ad}(x_1)\mathrm{Ad}(x_2)\dots\mathrm{Ad}(x_n)](y) = I_{\mathfrak{g}}$ and $x_1 * \dots * x_n \in \ker(H \rightarrow G_{\mathcal{D}}^{ad}(\mathbb{F}))$. Hence, ϕ is well-defined.

It remains to determine the kernel of ϕ . Suppose $y = x_1 x_2 \dots x_n \in \ker(\phi)$ with all x_i are either $Y_\alpha(\mathbf{s})$, or in T . Arguing as above, $\rho(z) = \rho(\phi(y)(z)) = y\rho(z)y^{-1}$ for all $z \in \mathfrak{g}$. So $y \in \mathrm{End}(V, \rho)$: it commutes with all $\rho(e_\alpha)$, hence with all $Y_\alpha(\mathbf{s})$. Since T acts faithfully, y commutes with T as well. Commuting with all generators of G_V , y is inevitably central. \square

As soon as there are few endomorphisms, the map ϕ in Theorem 1.2 can be “reversed” to define a projective representation of the Kac-Moody group.

Corollary 1.3. *Suppose that in the conditions of Theorem 1.2 the representation (V, ρ) is a brick, i.e., $\mathrm{End}(V, \rho) = \mathbb{F}$. Then*

$$\theta : G_{\mathcal{D}}(\mathbb{F}) \rightarrow \mathrm{GL}(V), \quad \theta(X_\alpha(\mathbf{s})) = Y_\alpha(\mathbf{s}), \quad \theta(t) = \widehat{\rho}(t) \quad \text{for } t \in T$$

extends to a projective representation of $G_{\mathcal{D}}(\mathbb{F})$.

If the root datum is simply-connected, i.e., α_i form a basis of \mathfrak{h} , then the group $G_{\mathcal{D}}(\mathbb{F})$ is generated by U_α -s [CtCh-93]. Hence, no grading is needed to define a representation of $G_{\mathcal{D}}(\mathbb{F})$, with all the proofs going through as before:

Corollary 1.4. *Suppose that \mathcal{D} is simply-connected and $p > \max_{i \neq j}(-A_{i,j})$. If (V, ρ) is a faithful, over-restricted brick for $\mathfrak{g}_{\mathbb{F}}$, then*

$$\theta : G_{\mathcal{D}}(\mathbb{F}) \rightarrow \mathrm{GL}(V), \quad \theta(X_\alpha(\mathbf{s})) = Y_\alpha(\mathbf{s})$$

extends to a projective representation of $G_{\mathcal{D}}(\mathbb{F})$.

2. REPRESENTATIONS OF COMPLETED GROUP

2.1. Completion. The group $G_{\mathcal{D}}(\mathbb{F})$ is also known as the “minimal” Kac-Moody group, while some of its various completions $\widehat{G_{\mathcal{D}}(\mathbb{F})}$ go under the name a “maximal” Kac-Moody group.

Let us consider a group G with a BN-pair (B, N) . Let \hat{G} be a completion of G with respect to some topology. Is (\overline{B}, N) (where \overline{B} is the closure of B in \hat{G}) a BN-pair on \hat{G} ? It depends on circumstances. For example, consider a simple split group scheme \mathcal{G} , $G = \mathcal{G}(\mathbb{F}_q[z, z^{-1}])$, the group of monomial matrices $N \leq G$ and positive and negative Iwahori subgroups $I_{\pm} = [\mathcal{G}(\mathbb{F}_q[z^{\pm 1}])] \xrightarrow{z^{\pm 1} \mapsto 0} \mathcal{G}(\mathbb{F}_q)]^{-1}(B)$. Both pairs (I_{\pm}, N) are BN-pairs on G but only $(\overline{I_+}, N)$ is a BN-pair on the positive completion $\hat{G} = \mathcal{G}(\mathbb{F}_q((z)))$: the countable groups $\hat{I_-} = I_-$ and N cannot generate uncountable \hat{G} . The following theorem pinpoints the completion process for groups with a BN-pair under some conditions:

Theorem 2.1. [CR-17, Th. 1.2] *Let G be a group with a BN-pair (B, N) with Weyl group (W, S) where S is finite. Suppose further that a topology \mathcal{T} on B is given such that the four conditions (1)–(4) hold.*

- (1) (B, \mathcal{T}) is a topological group.
- (2) The completion \hat{B} is a group.
- (3) $\mathcal{T}_1 := \{A \in \mathcal{T} \mid 1 \in A\}$ is a basis at 1 of topology on each minimal parabolic P_s , $s \in S$ that defines a structure of topological group on P_s .
- (4) The index $|P_s : B|$ is finite for each $s \in S$.

Under these conditions the following statements hold:

- (a) \mathcal{T}_1 is a basis at 1 of topology on G that defines a structure of topological group on G .
- (b) The completion \hat{G} is a group. The completion \hat{B} is equal to the closure \overline{B} .
- (c) The completion \hat{G} is isomorphic to the amalgam $\ast_{\mathfrak{B}} H$ where $\mathfrak{B} = \{\overline{B}, N, \overline{P_s}; s \in S\}$.
- (d) The pair (\overline{B}, N) is a BN-pair on the completed group \hat{G} .

Proof. It is a well-known theorem of Tits that a group with a BN-pair is an amalgam of N and its minimal parabolics [Ku-02, Prop. 5.1.7]. Later on Tits has shown how to back-engineer this group from such amalgam [T-81] (cf. [Ku-02, Th. 5.1.8]).

This is the heart of the proof: we pinpoint the completed group in part (c) but need to check numerous technical conditions of the Tits theorem. See [CR-17] for full details. \square

Theorem 2.1 gives us a locally pro- p -complete Kac-Moody group $G^{lpp} := \widehat{G_{\mathcal{D}}(\mathbb{F}_q)}$, $q = p^m$ by choosing the pro- p -topology on B : its basis at 1 is $\{A \leq B \mid |B : A| = p^a \text{ for some } a \in \mathbb{N}\}$. The Borel and the minimal parabolic subgroups of $G_{\mathcal{D}}(\mathbb{F}_p)$ are split [CaRé-09, 6.2] :

$$B = T \ltimes U_+, \quad P_s = L_s \ltimes U_s \text{ where } L_s = \langle U_{\alpha_i} \cup U_{-\alpha_i} \rangle T, \quad s = s_i, \quad U_s := U_+ \cap sU_+s^{-1}.$$

In particular, $|P_i : B|$ is finite for all $i \in I$ so that, by Theorem 2.1, we can complete $G_{\mathcal{D}}(\mathbb{F}_q)$ with respect to the pro- p -topology on B (or, in fact, any “ M_s -equivariant” topology). The group G^{lpp} has a BN-pair (\hat{B}, N) where $\hat{B} = H \ltimes \hat{U}_+$ and \hat{U}_+ is the full pro- p completion of U .

The congruence subgroup $C(G^{lpp}) = \cap_{g \in G^{lpp}} \widehat{U}_+^g$ is of crucial interest. Suppose that A is irreducible and the root datum \mathcal{D} is simply connected. Let $Z'(G^{lpp}) := Z((G_{\mathcal{D}}(\mathbb{F}_q) \times C(G^{lpp}))$ (note that the intersection is trivial).

Theorem 2.2. [CR-17, Ma-14, CrERi-08] *Under these conditions $G^{lpp}/Z'(G^{lpp})$ is a topologically simple group. Moreover, if A is 2-spherical, then $G^{lpp}/Z'(G^{lpp})$ is an abstractly simple group.*

2.2. Comparison to other completions. It is instructive to compare G^{lpp} with other completions of $G_{\mathcal{D}}(\mathbb{F}_q)$, a.k.a topological Kac-Moody groups (cf. [CR-17, Ma-13, Ro-16, CrERi-08, ReW-15]). Let us list them:

- the *Caprace-Rémy-Ronan group* G^{crr} , a completion in the topology of the action on Bruhat-Tits building,
- the *Carbone-Garland group* $G^{c\lambda}$, a completion in the topology of the action on the integrable simple module with a highest weight λ ,
- the *Mathieu-Rousseau group* G^{ma+} , an analogue of the ind-algebraic completion, also obtained as an amalgam $\ast_{\mathfrak{B}} H$ where $\mathfrak{B} = \{\tilde{B}, N, \tilde{P}_s\}$ with specially constructed groups \tilde{P}_s .
- another *Mathieu-Rousseau group* G^+ , the closure of G in G^{ma+} ,
- the *Belyaev group* G^b , the “largest” completion with compact totally disconnected \overline{U}_+ .
- the *Schlichting group* G^s , the “smallest” completion with compact totally disconnected \overline{U}_+ .

If $p > \max_{i \neq j} (-A_{ij})$, then $G^+ = G^{ma+}$ but they could be different, in general [Ro-16, 6.11]. The precise meaning of the “largest” and the “smallest” of the last two groups is a certain universal property (consult [ReW-15] for precise statement). The action on the Bruhat-Tits building ensures that $G^s = G^{crr}$. Theorem 2.1 gives G^b by considering the profinite topology on B instead of the pro- p -topology: its basis at 1 is $\{A \leq B \mid |B : A| < \infty\}$. The following theorem compares the known completions:

Theorem 2.3. [CR-17, Ro-16, ReW-15] *There are open continuous surjective group homomorphisms:*

$$G^b \twoheadrightarrow G^{lpp} \twoheadrightarrow G^+ \twoheadrightarrow G^{c\lambda} \twoheadrightarrow G^{crr} \xrightarrow{\cong} G^s.$$

2.3. Davis Building. Let G^* be one of the locally compact, totally disconnected groups from Section 2.2. It admits a BN-pair (B^*, N) with the same Weyl group (W, S) as the Kac-Moody algebra $\mathfrak{g}_{\mathbb{C}}$. Consequently, G^* acts on two simplicial complexes: the Bruhat-Tits building \mathfrak{B} and the *Davis building* \mathfrak{D} (also known as the Davis realisation [AB-08, Section 12.4] or the geometric realisation [D-08, Section 18.2]). Notice that there are variations in this definition: the original Davis’ definition produces a cell complex, while \mathfrak{D} (defined below) is a simplicial complex, a subdivision of this cell complex.

While the building \mathfrak{B} is well-known, it is still instructive to recall its definition. Let $\mathcal{P}(G^*)$ be the set of all proper parabolic subgroups of G^* . A parabolic $P \in \mathcal{P}(G^*)$ is conjugate to precisely one of the standard parabolics $P_J := \langle B^*, \dot{s} \rangle_{s \in J}$ where $J \subset S$, $\dot{s} \in G^*$ is a lift of the element $s \in W = N(T)/T$. Thus, we can define the *type* and the *rank* of each parabolic by

$$t(P) = J, \quad r(P) = |J| \quad \text{whenever } P \sim P_J.$$

The building \mathfrak{B} is an n -dimensional simplicial complex ($n = |S|$) whose set of k -dimensional simplices \mathfrak{B}_k is equal to $t^{-1}(n-k) = \{P \mid r(P) = n-k\}$. A simplex P' is a face of P if and only if $P \subseteq P'$. The group G^\star acts on \mathfrak{B} in the obvious way: ${}^gP = gPg^{-1}$. Since parabolic subgroups are self-normalising, the stabiliser of P is P itself. One drawback of this action is that stabilisers of simplices are not necessarily compact. This drawback is fixed in the Davis building.

A subset $J \subset S$ is called *spherical* if the Coxeter subgroup $\langle J \rangle$ is finite. Let $\mathcal{P}^{sp}(G^\star)$ be the subset of $\mathcal{P}(G^\star)$ that consists of parabolics of spherical type. The set $\mathcal{P}^{sp}(G^\star)$ is partially ordered under inclusion. The Davis building \mathfrak{D} is the geometric realisation of the poset $\mathcal{P}^{sp}(G^\star)$, i.e., its set \mathfrak{D}_k of k -dimensional simplices consists of $(k+1)$ -long chains of spherical parabolics

$$P_0 \subset P_1 \subset \dots \subset P_{k-1} \subset P_k.$$

Faces of a simplex are its subchains. The group G^\star acts on \mathfrak{D} in the same obvious way: $g(P_i) = (gP_i g^{-1})$. The stabiliser of a chain (P_i) is P_0 . Thus, all stabilisers are compact because spherical parabolic subgroups are necessarily compact.

One interesting example is a *generic* Kac-Moody group. Suppose $A_{i,j}A_{j,i} \geq 4$ for all i, j . Then the only spherical subsets of S are the empty set and one element subsets. Consequently, any chain in $\mathcal{P}^{sp}(G^\star)$ is of length at most 1 and \mathfrak{D} is a tree.

If \mathcal{A} has no irreducible components of finite type, both buildings \mathfrak{B} and \mathfrak{D} are contractible. If \mathcal{A} has an irreducible component of finite type, then \mathfrak{D} is still contractible, while \mathfrak{B} is not (see [D-08] for this as well as detailed study of \mathfrak{D}). We finish this section by stating Davis' Theorem:

Theorem 2.4. [D-04] *\mathfrak{D} admits a locally Euclidean, G^\star -invariant metric that turns \mathfrak{D} into complete, CAT(0) geodesic space.*

2.4. Projective Dimension of Smooth Representations. We study representations of G^\star over a field \mathbb{K} of characteristic zero. A representation (V, ρ) is called a *smooth representation* if for all $v \in V$ there exists a compact open subgroup $K_v \leq G^\star$ such that $\rho(g)v = v$ for all $g \in K_v$. Equivalently, the action $G \times V \rightarrow V$ is required to be continuous with respect to the discrete topology on V (and standard topologies in G^\star and the product).

The category $\mathcal{M}(H)$ of smooth representations of a locally compact totally disconnected topological group H is abelian with enough projectives [B-92, BuH-06]. In case of the group G^\star we can say more by examining its action on \mathfrak{D} :

Theorem 2.5. [HrR-17] *Let d be the dimension of \mathfrak{D} . Then*

$$\text{coh. dim}(\mathcal{M}(G^\star)) \leq d.$$

Proof. Let $C_i = C_i(\mathfrak{D}, \mathbb{K})$ be the group of \mathbb{K} -linear chains on \mathfrak{D} . The chain complex $\mathcal{C} = (C_d \xrightarrow{\partial} C_{d-1} \cdots \xrightarrow{\partial} C_0)$ is acyclic since \mathfrak{D} is contractible, i.e., all homology groups are trivial except for $H_0(\mathcal{C}) = \mathbb{K}$. This gives an exact sequence

$$0 \rightarrow C_d \xrightarrow{\partial} C_{d-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \rightarrow \mathbb{K} \rightarrow 0$$

of smooth representations of G^\star where \mathbb{K} is the trivial representation.

Let $\sigma = ((P_i), \tau)$ be an oriented simplex in \mathfrak{D} . Its stabiliser $\text{Stab}_{G^\star}(\sigma)$ is open: it is either P_0 , or its subgroup of index 2, depending on whether an element of G^\star can reverse the orientation τ or not. The one-dimensional space $\mathbb{K}[\sigma]$ is a smooth representation of P_0 . Since P is compact, $\mathbb{K}[\sigma]$ is projective in $\mathcal{M}(P_0)$. Since P_0

is open, the algebraic induction $\mathbb{K}G^\star \otimes_{\mathbb{K}P_0}$ is left adjoint to the restriction functor $\mathcal{M}(G^\star) \rightarrow \mathcal{M}(P_0)$. Hence, $\mathbb{K}G^\star \otimes_{\mathbb{K}P_0} \mathbb{K}[\sigma]$ is a projective module in $\mathcal{M}(G^\star)$. Observe that

$$C_m \cong \oplus_{(P_i)} \mathbb{K}G^\star \otimes_{\mathbb{K}P_0} \mathbb{K}[(P_i), \tau]$$

where the sum is taken over representatives of G^\star -orbits on \mathfrak{D}_m . It follows that \mathcal{C} is a projective resolution of the trivial representation \mathbb{K} .

Let V be an object in $\mathcal{M}(G^\star)$. Tensor product of representations $\otimes V$ is an exact functor $\mathcal{M}(G^\star) \rightarrow \mathcal{M}(G^\star)$ so that

$$0 \rightarrow C_d \otimes V \rightarrow C_{d-1} \otimes V \rightarrow \cdots \rightarrow C_0 \otimes V \rightarrow V \rightarrow 0$$

is an exact sequence. We claim that it is a projective resolution of V . Indeed, the functor $\mathcal{F} = \text{hom}(C_m, _)$ is exact since C_m is projective. The functor of all linear maps $\mathcal{E} = \text{hom}_{\mathbb{K}}(V, _)$ is also exact. The composition of two exact functors is exact, so $\mathcal{F}\mathcal{E} = \text{hom}(C_m \otimes V, _)$ is exact and $C_m \otimes V$ are projective objects. \square

2.5. Localisation. One should put Theorem 2.5 into a broader perspective of Schneider-Stuhler Localisation [HrR-17, ScSt-97]. By localisation we understand an equivalence of two categories: a representation theoretic category ($\mathcal{M}(G^\star)$ for us) is equivalent to (“localised to”) a geometric category. The key geometric category is the category $\text{Csh}_{G^\star}(\mathfrak{D})$ of G^\star -equivariant cosheaves on \mathfrak{D} .

A G^\star -equivariant cosheaf, a.k.a. a coefficient system for homology, is a datum $\mathcal{C} = (\mathcal{C}_F, r_{F'}^F, g_F)$ where \mathcal{C}_F is a \mathbb{K} -vector space for each face F of \mathfrak{D} , $r_{F'}^F : \mathcal{C}_F \rightarrow \mathcal{C}_{F'}$ is a linear map for each pair of faces $F' \subseteq F$, $g_F : \mathcal{C}_F \rightarrow \mathcal{C}_{gF}$ is a linear map for all $g \in G^\star$ and a face F that are subject to the following axioms:

- (i) $r_F^F = \text{id}_F$ for every face F ,
- (ii) $r_{F''}^{F'} \circ r_{F'}^F = r_{F''}^F$ for faces $F'' \subseteq F' \subseteq F$,
- (iii) $g_{hF} \circ h_F = (gh)_F$ for all g, h and F ,
- (iv) \mathcal{C}_F is a smooth representation of the stabiliser G_F^\star for all F ,

$$\begin{array}{ccc} \mathcal{C}_F & \xrightarrow{g_F} & \mathcal{C}_{gF} \\ \text{(iii) The square} & \begin{array}{ccc} \downarrow r_{F'}^F & & \downarrow r_{gF'}^{gF} \\ \mathcal{C}_{F'} & \xrightarrow{g_{F'}} & \mathcal{C}_{gF'} \end{array} & \text{is commutative for all } g \text{ and } F' \subseteq F. \end{array}$$

A morphism of equivariant cosheaves $\psi : \mathcal{C} \rightarrow \mathcal{E}$ is a system of linear maps $\psi_F : \mathcal{C}_F \rightarrow \mathcal{E}_F$, commuting with actions and restrictions, i.e., the squares

$$\begin{array}{ccc} \mathcal{C}_F & \xrightarrow{\psi_F} & \mathcal{E}_F \\ \downarrow r_{F'}^F & & \downarrow r_{F'}^F \\ \mathcal{C}_{F'} & \xrightarrow{\psi_{F'}} & \mathcal{E}_{F'} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{C}_F & \xrightarrow{\psi_F} & \mathcal{E}_F \\ \downarrow g_F & & \downarrow g_F \\ \mathcal{C}_{gF} & \xrightarrow{\psi_{gF}} & \mathcal{E}_{gF} \end{array}$$

are commutative for all g and $F' \subseteq F$.

The category of equivariant cosheaves $\text{Csh}_{G^\star}(\mathfrak{D})$ is an abelian category [ScSt-97]: kernels and cokernels can be computed simplexwise. There are several functors connecting the key categories $\mathcal{M}(G^\star)$ and $\text{Csh}_{G^\star}(\mathfrak{D})$. For instance, *the trivial cosheaf functor* \mathcal{L} associates a cosheaf $\underline{V} \in \text{Csh}_{G^\star}(\mathfrak{D})$ to $(V, \rho) \in \mathcal{M}(G^\star)$:

$$\underline{V}_F = V, \quad r_{F'}^F = \text{id}_V, \quad g_F = \rho(g).$$

In the opposite direction, if \mathcal{C} is a G^\star -equivariant cosheaf, the group G^\star acts on the vector space of oriented i -chains (with finite support) $C_i(\mathfrak{D}, \mathcal{C})$ with coefficients in

\mathcal{C} . In fact, G^\star acts on the space of more general chains as well but the finite support ensures that $C_i(\mathfrak{D}, \mathcal{C}) \in \mathcal{M}(G^\star)$, a functor in the opposite direction! Furthermore, the chain complex

$$\mathcal{C}(\mathcal{C}) : 0 \rightarrow C_d(\mathfrak{D}, \mathcal{C}) \xrightarrow{\partial} C_{d-1}(\mathfrak{D}, \mathcal{C}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(\mathfrak{D}, \mathcal{C}) \rightarrow 0$$

is a chain complex in $\mathcal{M}(G^\star)$. The functor \mathcal{C} allows us to paraphrase Theorem 2.5:

Corollary 2.6. *Given a smooth representation V , the complex $\mathcal{C}(\underline{V})$ is a projective resolution of V .*

Resolutions of the form $\mathcal{C}(\mathcal{C})$ are quite useful. The category $\text{Csh}_{G^\star}(\mathfrak{D})$ is *Noetherian*: a subobject of a finitely-generated object is finitely-generated. Hence, a finitely-generated infinite-dimensional object V admits a finitely-generated projective resolution, yet $\mathcal{C}(\underline{V})$ is not finitely generated. We call a finitely-generated projective resolution of the form $\mathcal{C}(\mathcal{C})$ a *Schneider-Stuhler resolution*. Do they exist (cf. Section 3.8)?

It may be possible to construct them using systems of subgroups or system of idempotents [HrR-17, MeS-10]. In fact, [HrR-17] contains a positive answer to existence of Schneider-Stuhler resolutions modulo a (yet open) conjecture on homology of a CAT(0)-complex. To satisfy the reader's curiosity we state this conjecture in full in Section 3.9.

Let us turn our attention to the localisation. We have the trivial cosheaf and the 0-th homology functors going between $\mathcal{M}(G^\star)$ and $\text{Csh}_{G^\star}(\mathfrak{D})$:

$$\mathcal{L}((V, \rho)) = \underline{V}, \quad \mathcal{H}(\mathcal{C}) = H_0(\mathfrak{D}, \mathcal{C}).$$

Let $\Sigma \subset \text{Mor}(\text{Csh}_{G^\star}(\mathfrak{D}))$ be the class of morphisms ψ such that $\mathcal{H}(\psi)$ is an isomorphism. Consider the category of left fractions $\text{Csh}_{G^\star}(\mathfrak{D})[\Sigma^{-1}]$ and the fraction functor $\mathcal{Q}_\Sigma : \text{Csh}_{G^\star}(\mathfrak{D}) \rightarrow \text{Csh}_{G^\star}(\mathfrak{D})[\Sigma^{-1}]$. Note that while these fractions always exist, Σ needs to satisfy *the left Ore condition* (a.k.a. admit a calculus of left fractions) for these objects to be malleable [GZ-67]. The 0-th cohomology functor extends to a functor from the category of left fractions $\mathcal{H}[\Sigma^{-1}] : \text{Csh}_{G^\star}(\mathfrak{D})[\Sigma^{-1}] \rightarrow \mathcal{M}(G^\star)$. We are ready for the localisation theorem, a generalisation of Schneider-Stuhler Localisation [ScSt-97]:

Theorem 2.7. [HrR-17] *Under the notations established above, the following statements hold:*

- (i) *The class Σ satisfies the left Ore condition.*
- (ii) *The functor $\mathcal{H}[\Sigma^{-1}] : \text{Csh}_{G^\star}(\mathfrak{D})[\Sigma^{-1}] \rightarrow \mathcal{M}(G^\star)$ is an equivalence of categories.*
- (iii) *$\mathcal{Q}_\Sigma \circ \mathcal{L}$ is a quasi-inverse of $\mathcal{H}[\Sigma^{-1}]$.*

3. QUESTIONS

3.1. Isomorphism Problem. Find necessary and sufficient conditions on realisations \mathcal{R} and \mathcal{S} for $\hat{L}_{\mathcal{R}}$ and $\hat{L}_{\mathcal{S}}$ to be equivalent as functors to graded Lie algebras.

3.2. Existence of Restricted Structure. Suppose \mathbb{F} is a field of positive characteristic. Find necessary and sufficient conditions on realisation for $\hat{L}_{\mathcal{R}}(\mathbb{F})$ to admit a structure of restricted Lie algebra. In particular, if \mathcal{A} is a generalised Cartan matrix of general type and $p \leq \max_{i \neq j} (-A_{i,j})$, could $\hat{L}_{\mathcal{D}}(\mathbb{F})$ be restricted?

3.3. Humphreys-Verma Conjecture. Consider “natural” restricted $\mathfrak{g}_{\mathbb{F}}$ -modules, e.g., irreducible, projective, injective. Do they admit an action of $G_{\mathcal{D}}(\mathbb{F})$ such that for each real root α the differential of the U_{α} -action is the $\mathfrak{g}_{\mathbb{F}\alpha}$ -action?

3.4. Theory of Over-restricted Representations. Investigate algebraic properties of the over-restricted enveloping algebra $U(\mathfrak{g}_{\mathbb{F}})/(x^p - x^{[p]}, e_{\alpha}^{[(p+1)/2]})$ and its representations.

3.5. Congruence Kernel. Develop techniques for computing $C(G^{lpp})$. Find necessary and sufficient conditions for $C(G^{lpp})$ to be trivial (central, finitely pro- p -generated, etc.).

3.6. Lattices in Locally Pro- p -complete Kac-Moody Groups. Find minimal covolume of lattices (uniform and overall) in G^{lpp} .

3.7. Completions. Investigate the completions. Find necessary and sufficient conditions the following completions to be equal $G^b \stackrel{?}{=} G^{lpp}$, $G^+ \stackrel{?}{=} G^{ma+}$, $G^{c\lambda} \stackrel{?}{=} G^{crr}$.

3.8. Schneider-Stuhler Resolution. Does a Schneider-Stuhler resolution exist for any finitely-generated object $V \in \mathcal{M}(G^{\star})$? What about irreducible objects? More precisely, does there exist a family of functors $\mathcal{T}_k : \mathcal{M}(G^{\star}) \rightarrow \text{Csh}_{G^{\star}}(\mathfrak{D})$, indexed by natural numbers, such that for each irreducible $L \in \mathcal{M}(G^{\star})$ there exists $N \in \mathbb{N}$ such that $\mathcal{C}(\mathcal{T}_k(L))$ is a Schneider-Stuhler resolution of L for all $k > N$.

3.9. Homology of CAT(0)-Complex. Let \mathfrak{X} be a CAT(0)-simplicial complex, A an abelian group. Suppose we have an idempotent operator $\Lambda_x : A \rightarrow A$ for each vertex x of \mathfrak{X} . We call this system of idempotents *geodesic* if the following conditions hold:

- (i) $\Lambda_x \Lambda_y = \Lambda_y \Lambda_x$ if x and y are adjacent,
- (ii) $\Lambda_x \Lambda_z \Lambda_y = \Lambda_x \Lambda_y$ and $\Lambda_x \Lambda_z = \Lambda_z \Lambda_x$ if z is any vertex of the first simplex along the geodesic $[x, y]$ for all vertices x and y .

Such geodesic system gives a cosheaf \underline{A}_{Λ} where \underline{A}_{Λ} is the image of the product $\prod_x \Lambda_x$ taken over all faces of F and $r_{F'}^F$ are natural inclusions.

Is it true that $H_m(\mathcal{X}, \underline{A}_{\Lambda}) = 0$ for all $m > 0$?

A positive answer to this question for Bruhat-Tits buildings can be obtained by the methods of Meyer and Solleveld [MeS-10].

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